Nonnormality of the Stoneham Constants

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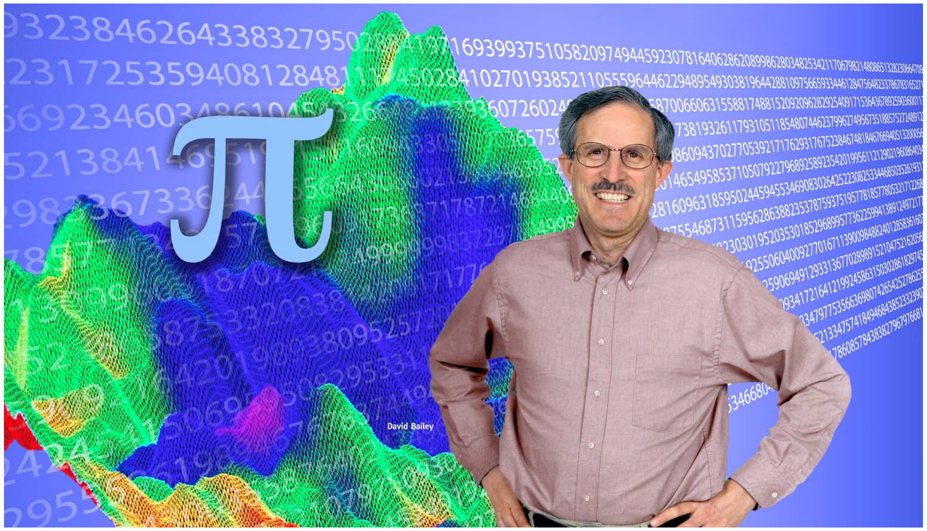


Normal numbers

Given an integer $b > 1$, a real number x is ***b*-normal** (or "normal base b") if every m-long string of digits in the base-b expansion of x appears with precisely the expected limiting frequency b^{-m}.

Using measure theory, it is can be shown that almost all real numbers are b-normal for a given integer base b. In fact, almost all reals are b-normal for all integer bases $b > 1$ simultaneously (i.e., are "absolutely normal").

These are widely believed to be b-normal, for all integer bases $b > 1$:

$\pi = 3.1415926535\ldots$

e = 2.7182818284…

sqrt(2) = 1.4142135623…

log(2) = 0.6931471805…

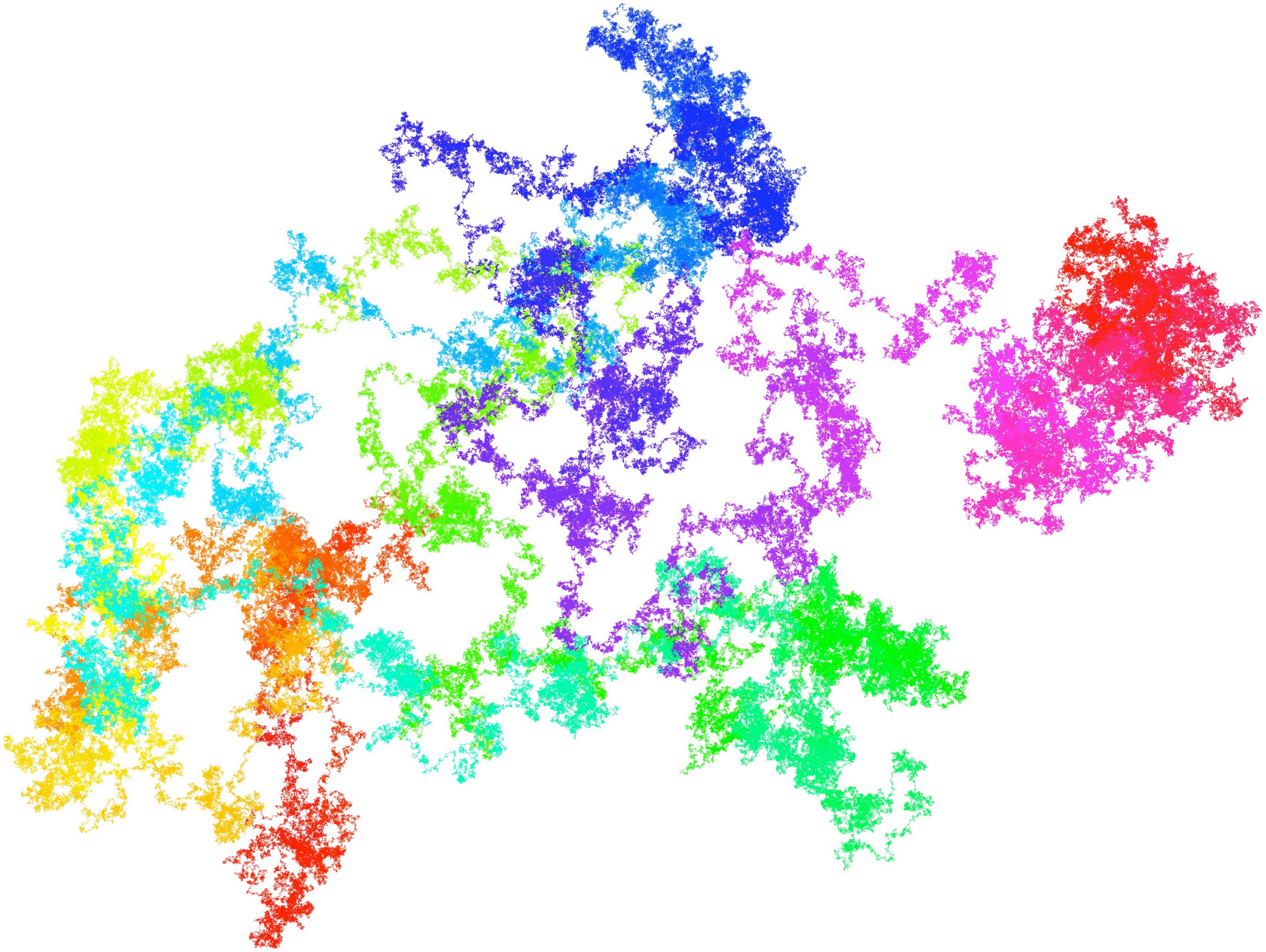
Every irrational algebraic number (this conjecture is due to Borel).

But there are no normality proofs for any of these constants in any base, nor are there any nonnormality results for any of these constants.

Until recently, normality proofs were known only for a few relatively contrived examples such as Champernowne's constant = 0.12345678910111213141 5… (which is 10-normal).

There are no proofs of absolute normality for *any* explicit constant.

Random walk on the first two billion bits of pi



DHB, J. M. Borwein, C. S. Calude, M. J. Dinneen, M. Dumitrescu and A. Yee, “An Empirical Approach to the Normality of Pi,” manuscript, 26 Nov 2011, available athttp://crd.lbl.gov/~dhbailey/dhbpapers/normality.pdf. This also includes numerous statistical analyses of the digits of pi.

A result for algebraic numbers

If x is algebraic of degree $d > 1$, then its binary expansion through position n must have at least $C\, n^{1/d}$ 1-bits, for all sufficiently large n and some C that depends on x.

Example: The first n binary digits of sqrt(2) must have at least sqrt(n) ones.

In this example, the result follows by noting that the one-bit count of the product of two integers is less than or equal to the product of the one-bit counts of the two integers. The more general result above requires a more sophisticated approach.

However, note that these results are still a far cry from full normality.

DHB, J. M. Borwein, R. E. Crandall and C. Pomerance, “On the Binary Expansions of Algebraic Numbers,” *Journal of Number Theory Bordeaux*, vol. 16 (2004), pg. 487-518.

Pseudorandom sequences and the normality of log 2 and pi

Consider the sequence $x_0 = 0$, and

$$x_n = \left(2x_{n-1} + \frac{1}{n}\right) \bmod 1$$

If it can be demonstrated that this sequence is equidistributed in the unit interval, then this would imply that log 2 is 2-normal.

Similarly, consider the sequence $x_0 = 0$, and

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

If it can be shown that this sequence is equidistributed in the unit interval, then this would imply that pi is 16-normal (and hence 2-normal).

DHB and R. E. Crandall, "On the Random Character of Fundamental Constant Expansions," *Experimental Mathematics*, vol. 10, no. 2 (Jun 2001), pg. 175-190.

A class of provably normal constants

DHB and Richard Crandall have shown that an infinite class of constants is *b*-normal (and thus b^m-normal for any positive integer *m*), for instance:

$$\begin{aligned} \alpha_{2,3} &= \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n}} \\ &= 0.041883680831502985071252898624571682426096\ldots_{10} \\ &= 0.0ab8e38f684bda12f684bf35ba781948b0fcd6e9e0\ldots_{16} \end{aligned}$$

This particular constant was proven 2-normal by Stoneham in 1971. We extended this to the case where (2,3) are any pair (*b*,*c*) of relatively prime integers > 1, and also to an uncountable class (here r_n is *n*-th bit of *r* in [0,1)):

$$\alpha_{2,3}(r) = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n + r_n}}$$

More recently, DHB and Michal Misiurewich established the $\alpha_{2,3}$ result more simply by means of a “hot spot” lemma proved using ergodic theory.

DHB and M. Misiurewicz, “A Strong Hot Spot Theorem," *Proceedings of the American Mathematical Society*, vol. 134 (2006), no. 9, pg. 2495-2501.
DHB and R. E. Crandall, “Random Generators and Normal Numbers,” *Experimental Mathematics*, vol. 11, no. 4 (2002), pg. 527-546.

A nonnormality result

Although $\alpha_{2,3}$ is provably 2-normal, surprisingly enough it is NOT 6-normal. Note that we can write

$$6^n \alpha_{2,3} \bmod 1 = \left(\sum_{m=1}^{\lfloor \log_3 n \rfloor} 3^{n-m} 2^{n-3^m} \right) \bmod 1 + \sum_{m=\lfloor \log_3 n \rfloor + 1}^{\infty} 3^{n-m} 2^{n-3^m}$$

The first portion of this expression is zero, since all of the terms in the summation are integers. When $n = 3^m$, the second portion is accurately approximated by the first term of the series. Thus,

$$6^{3^m} \alpha_{2,3} \bmod 1 \approx \frac{\left(\frac{3}{4}\right)^{3^m}}{3^{m+1}}$$

Because this is so small for large m, this means the base-6 expansion of $\alpha_{2,3}$ has long stretches of zeroes beginning at positions $3^m + 1$. This observation can be fashioned into a rigorous proof of nonnormality.

DHB and J. M. Borwein, "Normal Numbers and Pseudorandom Generators," available at http://crd.lbl.gov/~dhbailey/dhbpapers/normal-pseudo.pdf.

$\alpha_{2,3}$ in base-6 digits

0.
013014043000333425113050213000000124355504543223301150024352532055135234354101
0430000000000000000514113005404055545530314425043343510124134523511251421 25134
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024511252554525045245533541442124051204242044021002315121441215553532044450502
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A general nonnormality result for the Stoneham alpha constants

Given co-prime integers $b > 1$ and $c > 1$, and integers $p, q, r > 0$, with neither b nor c dividing r, let $B = b^p c^q r$, and assume this condition holds:

$$D = c^{q/p} r^{1/p} / b^{c-1} < 1$$

Then the constant

$$\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$$

is not B-normal. Thus, for example, $\alpha_{b,c}$ is b-normal but not bc-normal.

It is not known whether or not this is a complete catalog of the bases for which a Stoneham constant is nonnormal.

Ref: DHB, "Nonnormality of the Stoneham constants, manuscript, 8 Dec 2011, available at http://crd.lbl.gov/~dhbailey/dhbpapers/nonnormality.pdf.

Questions

- There are many cases still not covered -- e.g., is $\alpha_{2,3}$ 3-normal, or not?
- Can we prove normality for sums of alphas, i.e. $\alpha_{2,3} + \alpha_{2,5}$?
  - This may be possible, although quite difficult – we are working on this now.
- Can we prove nonnormality for sums of alphas, i.e., $\alpha_{2,3} + \alpha_{3,2}$?
  - Yes – we are working on this now.
- Are there any “relatives” of the Stoneham constants that permit similar proofs of normality or nonnormality?
- Can absolute normality (i.e., *b*-normal for all integer bases *b* > 1 simultaneously) be established for ANY explicit constant? Almost all real numbers are absolutely normal, but there are no explicit examples, to our knowledge.
- Can normality (to any base) be established for any of the “natural” irrational constants of mathematics – e, pi, sqrt(2), log(2), zeta(3), etc.?
- For that matter, can nonnormality be proven for any of the natural irrational constants?